't Hooft GR 12.2

For massive particle, we have

$$\left(\frac{du}{d\varphi}\right)^2 \cancel{\varphi} = \frac{E^2-1}{J^2} + \frac{2M}{J^2}u - u^2 + 2Mu^3$$

In Newtonian Limit, we neglect u^3 term:

$$\left(\frac{du}{d\varphi}\right)^2 \cancel{\varphi} = \frac{E^2-1}{J^2} + \frac{2M}{J^2}u - u^2$$

$$\cancel{2u'}\,u'' = \frac{\cancel{2}M\cancel{u'}}{J^2} - \cancel{2}u\cancel{u'}$$

$$u'' = \frac{M}{J^2} - u.$$

E&M: $\quad u'' = -u + \frac{M}{J^2}$

Solved by $\quad u = A\cos[\varphi] + \frac{M}{J^2}$, or

$$\frac{1}{r} = A\cos[\varphi] + \frac{M}{J^2}$$

Light Ray enter at $r = \infty$, or $u = \frac{1}{r} = 0$

$$\Rightarrow \quad u[\varphi^*] = A\cos[\varphi^*] + \frac{M}{J^2} = 0.$$

$$A\cos[\varphi^*] = -\frac{M}{J^2},$$

$$\cos[\varphi^*] = -\frac{M}{AJ^2}$$

This clearly suggests an expansion around $\frac{\pi}{2}$

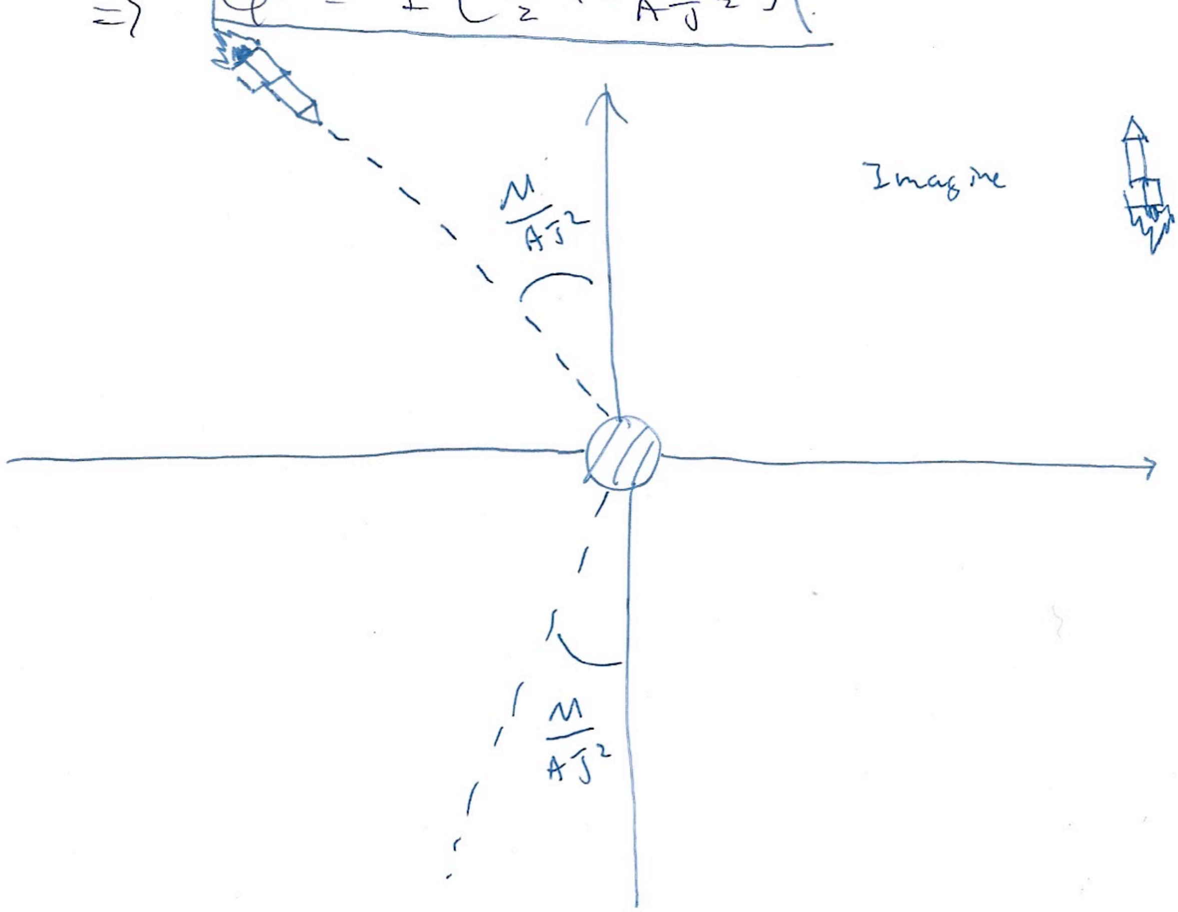
$$\cancel{\cos[\varphi]=}\ \cos[\tfrac{\pi}{2}+\delta\varphi] = \cos[\tfrac{\pi}{2}] - \sin[\tfrac{\pi}{2}]\,\delta\varphi$$

$$= -\delta\varphi$$

$$\Rightarrow \delta\varphi = \frac{M}{AJ^2}$$

$\Rightarrow \quad \varphi^* = \frac{\pi}{2} + \frac{M}{AJ^2}$, because $\cos$ is even, we have the closest other solution at $-\left(\frac{\pi}{2}+\frac{M}{AJ^2}\right)$.

$$\Rightarrow \boxed{\varphi^* = \pm\left(\frac{\pi}{2}+\frac{M}{AJ^2}\right)}$$



Imagine is light ray.

$\Rightarrow \quad \Delta = 2\frac{M}{AJ^2}$, Equating AJ^2 with r_0, it is $2\frac{M}{r_0}$, half of $4\frac{M}{r_0}$

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7-14-2024